

# The Gaussian coefficient revisited

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## Abstract

We give a new  $q$ -( $1+q$ )-analogue of the Gaussian coefficient, also known as the  $q$ -binomial which, like the original  $q$ -binomial  $\begin{bmatrix} n \\ k \end{bmatrix}_q$ , is symmetric in  $k$  and  $n-k$ . We show this  $q$ -( $1+q$ )-binomial is more compact than the one discovered by Fu, Reiner, Stanton and Thiem. Underlying our  $q$ -( $1+q$ )-analogue is a Boolean algebra decomposition of an associated poset. These ideas are extended to the Birkhoff transform of any finite poset. We end with a discussion of higher analogues of the  $q$ -binomial.

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## 1 Introduction

Inspired by work of Fu, Reiner, Stanton and Thiem [2], Cai and Readdy [1] asked the following question. Given a combinatorial  $q$ -analogue

$$X(q) = \sum_{w \in X} q^{a(w)},$$

where  $X$  is a set of objects and  $a(\cdot)$  is a statistic defined on the elements of  $X$ , when can one find a smaller set  $Y$  and two statistics  $s$  and  $t$  such that

$$X(q) = \sum_{w \in Y} q^{s(w)} \cdot (1+q)^{t(w)}.$$

Such an interpretation is called an  $q$ -( $1+q$ )-analogue. Examples of  $q$ -( $1+q$ )-analogues have been determined for the  $q$ -binomial by Fu, Reiner, Stanton and Thiem [2], and for the  $q$ -Stirling numbers of the first and second kinds by Cai and Readdy [1], who also gave poset and homotopy interpretations of their  $q$ -( $1+q$ )-analogues.

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In 1916 MacMahon [3, 4, 5] observed that the Gaussian coefficient, also known as the  $q$ -binomial coefficient, is given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \Omega_{n,k}} q^{\text{inv}(w)}.$$

Here  $\Omega_{n,k} = \mathfrak{S}(0^{n-k}, 1^k)$  denotes all permutations of the multiset  $\{0^{n-k}, 1^k\}$ , that is, all words  $w = w_1 \cdots w_n$  of length  $n$  with  $n - k$  zeroes and  $k$  ones, and  $\text{inv}(\cdot)$  denotes the inversion statistic defined by  $\text{inv}(w_1 w_2 \cdots w_n) = |\{(i, j) : 1 \leq i < j \leq n, w_i > w_j\}|$ . Fu et al. defined a subset  $\Omega'_{n,k} \subseteq \Omega_{n,k}$  and two statistics  $a$  and  $b$  such that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{w \in \Omega'_{n,k}} q^{a(w)} \cdot (1 + q)^{b(w)}.$$

In this paper we will return to the original study by Fu et al. of the Gaussian coefficient. We discover a more compact  $q$ -( $1 + q$ )-analogue which, like the original Gaussian coefficient, is also symmetric in the variables  $k$  and  $n - k$ . See Corollary 2.6 and Theorem 3.6. This symmetry was missing in Fu et al.'s original  $q$ -( $1 + q$ )-analogue. We give a Boolean algebra decomposition of the related poset  $\Omega_{n,k}$ . Since this poset is a distributive lattice, in the last section we extend these ideas to poset decompositions of any distributive lattice and other analogues.

## 2 A poset interpretation

In this section we consider the poset structure on 0-1-words in  $\Omega_{n,k}$ . For further poset terminology and background, we refer the reader to [6].

We begin by making the set of elements  $\Omega_{n,k}$  into a graded poset by defining the cover relation to be

$$u \circ 01 \circ v \prec u \circ 10 \circ v,$$

where  $\circ$  denotes concatenation of words. The word  $0^{n-k}1^k$  is the minimal element and the word  $1^k0^{n-k}$  is the maximal element in the poset  $\Omega_{n,k}$ . Furthermore, this poset is graded by the inversion statistic. This poset is simply the interval  $[\hat{0}, x]$  of Young's lattice, where the minimal element  $\hat{0}$  is the empty Ferrers diagram and  $x$  is the Ferrers diagram consisting of  $n - k$  columns and  $k$  rows.

An alternative description of the poset  $\Omega_{n,k}$  is that it is isomorphic to the Birkhoff transform of the Cartesian product of two chains. Let  $C_m$  denote the  $m$ -element chain. The poset  $\Omega_{n,k}$  is isomorphic to the distributive lattice of all lower order ideals of the product  $C_{n-k} \times C_k$ , usually denoted by  $J(C_{n-k} \times C_k)$ .

**Definition 2.1.** Let  $\Omega''_{n,k}$  consist of all 0,1-words  $v = v_1 v_2 \cdots v_n$  in  $\Omega_{n,k}$  such that

$$v_1 \leq v_2, \quad v_3 \leq v_4, \quad \dots, \quad v_{2 \cdot \lfloor n/2 \rfloor - 1} \leq v_{2 \cdot \lfloor n/2 \rfloor}.$$

Observe that when  $n$  is odd there is no condition on the last entry  $w_n$ . Define two maps  $\phi$  and  $\psi$  on  $\Omega_{n,k}$  by sending the word  $w = w_1 w_2 \cdots w_n$  to

$$\begin{aligned} \phi(w) &= \min(w_1, w_2), \max(w_1, w_2), \min(w_3, w_4), \max(w_3, w_4), \dots, \\ \psi(w) &= \max(w_1, w_2), \min(w_1, w_2), \max(w_3, w_4), \min(w_3, w_4), \dots \end{aligned}$$

The map  $\phi$  sorts the entries in positions 1 and 2, 3 and 4, and so on. If  $n$  is odd, the entry  $w_n$  remains in the same position. Similarly, the map  $\psi$  sorts in reverse order each pair of positions. Note that the map  $\phi$  maps  $\Omega_{n,k}$  surjectively onto the set  $\Omega''_{n,k}$ .

We have the following Boolean algebra decomposition of the poset  $\Omega_{n,k}$ .

**Theorem 2.2.** *The distributive lattice  $\Omega_{n,k}$  has the Boolean algebra decomposition*

$$\Omega_{n,k} = \bigcup_{v \in \Omega''_{n,k}} [v, \psi(v)].$$

*Proof.* Observe that the maps  $\phi$  and  $\psi$  satisfy the inequalities  $\phi(w) \leq w \leq \psi(w)$ . Furthermore, the fiber of the map  $\phi : \Omega_{n,k} \rightarrow \Omega''_{n,k}$  is isomorphic to a Boolean algebra, that is,  $\phi^{-1}(v) \cong [v, \psi(v)]$ .  $\square$

For  $v \in \Omega''_{n,k}$  define the statistic

$$\text{asc}_{\text{odd}}(v) = |\{i : v_i < v_{i+1}, i \text{ odd}\}|,$$

that is,  $\text{asc}_{\text{odd}}(\cdot)$  enumerates the number of ascents in odd positions.

**Corollary 2.3.** *The  $q$ -binomial is given by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{v \in \Omega''_{n,k}} q^{\text{inv}(v)} \cdot (1+q)^{\text{asc}_{\text{odd}}(v)}. \quad (2.1)$$

*Proof.* It is enough to observe that the sum of the inversion statistic over the elements in the fiber  $\phi^{-1}(v) = [v, \psi(v)]$  for  $v \in \Omega''_{n,k}$  is given by  $q^{\text{inv}(v)} \cdot (1+q)^{\text{asc}_{\text{odd}}(v)}$ .  $\square$

A geometric way to understand this  $q(1+q)$ -interpretation is to consider lattice paths from the origin  $(0,0)$  to  $(n-k, k)$  which only use east steps  $(1,0)$  and north steps  $(0,1)$ . Color the squares of this  $(n-k) \times k$  board as a chessboard, where the square incident to the origin is colored white. The map  $\phi$  in the proof of Theorem 2.2 corresponds to taking a lattice path where every time there is a north step followed by an east step that turns around a white square, we exchange these two steps. The statistic  $\text{asc}_{\text{odd}}$  enumerates the number of times an east step is followed by a north step when this pair of steps borders a white square.

Let  $\text{er}(n, k)$  denote the cardinality of the set  $\Omega''_{n,k}$ . Then we have

**Proposition 2.4.** *The cardinalities  $\text{er}(n, k)$  satisfy the recursion*

$$\text{er}(n, k) = \text{er}(n-2, k-2) + \text{er}(n-2, k-1) + \text{er}(n-2, k) \quad \text{for } 0 \leq k \leq n \text{ and } n \geq 2,$$

with the boundary conditions  $\text{er}(0,0) = \text{er}(1,0) = \text{er}(1,1) = 1$  and  $\text{er}(n, k) = 0$  whenever  $k > n$ ,  $k < 0$  or  $n < 0$ .

*Proof.* A word in  $\Omega''_{n,k}$  begins with either 00, 01 or 11, yielding the three cases of the recursion.  $\square$

Directly we obtain the generating polynomial.

**Theorem 2.5.** *The generating polynomial for  $\text{er}(n, k)$  is given by*

$$\sum_{k=0}^n \text{er}(n, k) \cdot x^k = (1 + x + x^2)^{\lfloor n/2 \rfloor} \cdot (1 + x)^{n - 2 \cdot \lfloor n/2 \rfloor}.$$

We end with a statement concerning the symmetry of the  $q$ -( $1 + q$ )-binomial.

**Corollary 2.6.** *The set of defining elements for the  $q$ -( $1 + q$ )-binomial satisfy the following symmetric relation:*

$$|\Omega''_{n,k}| = |\Omega''_{n,n-k}|.$$

*Proof.* This follows from the fact that the generating polynomial for  $\text{er}(n, k)$  is a product of palindromic polynomials, and thus is itself is a palindromic polynomial.  $\square$

### 3 Analysis of the Fu–Reiner–Stanton–Thiem interpretation

A *weak partition* is a finite non-decreasing sequence of non-negative integers. A weak partition  $\lambda = (\lambda_1, \dots, \lambda_{n-k})$  with  $n - k$  parts and each part at most  $k$  where  $\lambda_1 \leq \dots \leq \lambda_{n-k}$  corresponds to a Ferrers diagram lying inside an  $(n - k) \times k$  rectangle with column  $i$  having height  $\lambda_i$ . These weak partitions are in direct correspondence with the set  $\Omega_{n,k}$ .

Fu, Reiner, Stanton and Thiem used a pairing algorithm to determine a subset  $\Omega'_{n,k} \subseteq \Omega_{n,k}$  of 0-1-sequences to define their  $q$ -( $1 + q$ )-analogue of the  $q$ -binomial; see [2, Proposition 6.1]. This translates into the following statement. The set  $\Omega'_{n,k}$  is in bijection with weak partitions into  $n - k$  parts with each part at most  $k$  such that

- (a) if  $k$  is even, each odd part has even multiplicity,
- (b) if  $k$  is odd, each even part (including 0) has even multiplicity.

**Definition 3.1.** *Let  $\text{fst}(n, k)$  be the cardinality of the set  $\Omega'_{n,k}$ .*

**Lemma 3.2.** *The quantity  $\text{fst}(n, k)$  counts the number of weak partitions into  $n - k$  parts where each part is at most  $k$  and each odd part has even multiplicity.*

*Proof.* When  $k$  is even there is nothing to prove. When  $k$  is odd, by considering the complement of weak partitions with respect to the rectangle of size  $(n - k) \times k$ , we obtain a bijective proof. The same complement proof also shows the case when  $k$  is even holds.  $\square$

**Theorem 3.3.** *The first-coefficients satisfy the recursion*

$$\begin{aligned} \text{fst}(n, k) &= \text{fst}(n - 1, k - 1) + \text{fst}(n - 1, k) && \text{for } k \text{ even,} \\ \text{fst}(n, k) &= \text{fst}(n - 2, k - 2) + \text{fst}(n - 2, k - 1) + \text{fst}(n - 2, k) && \text{for } k \text{ odd,} \end{aligned}$$

where  $0 \leq k \leq n$  and  $n \geq 2$  with the boundary conditions  $\text{fst}(0, 0) = \text{fst}(1, 0) = \text{fst}(1, 1) = 1$  and  $\text{fst}(n, k) = 0$  whenever  $k > n$ ,  $k < 0$  or  $n < 0$ .

*Proof.* We use the characterization in Lemma 3.2. When  $k$  is even there are two cases. If the last part is  $k$ , remove it to obtain a weak partition counted by  $\text{frst}(n-1, k)$ . If the last part is less than  $k$ , then the weak partition is counted by  $\text{frst}(n-1, k-1)$ .

When  $k$  is odd there are three cases. If the last two parts are equal to  $k$ , then removing these two parts yields a weak partition counted by  $\text{frst}(n-2, k)$ . Note that we cannot have the last part equal to  $k$  and the next to last part less than  $k$  since  $k$  is odd. If the last part is equal to  $k-1$ , we can remove it to obtain a weak partition counted by  $\text{frst}(n-2, k-1)$ . Finally, if the last part is less than or equal to  $k-2$ , the weak partition is counted by  $\text{frst}(n-2, k-2)$ .  $\square$

**Remark 3.4.** For  $k$  odd we have the shorter recursion  $\text{frst}(n, k) = \text{frst}(n-1, k-1) + \text{frst}(n-2, k)$ . However, we use the longer recursion in the proof of Theorem 3.6.

**Lemma 3.5.** *The inequality  $\text{frst}(n, k) \leq \text{frst}(n+1, k+1)$  holds.*

*Proof.* The weak partitions which lie inside the rectangle  $(n-k) \times k$  and satisfy the conditions of Lemma 3.2 are included among the weak partitions which lie inside the larger rectangle  $(n-k) \times (k+1)$  and satisfy the same conditions.  $\square$

**Theorem 3.6.** *For all  $0 \leq k \leq n$  the inequality  $|\Omega''_{n,k}| = \text{er}(n, k) \leq \text{frst}(n, k) = |\Omega'_{n,k}|$  holds.*

*Proof.* We proceed by induction on  $n$ . The induction base is  $n \leq 3$ . Furthermore, the inequality holds when  $k$  is 0, 1,  $n-1$  and  $n$ . When  $k$  is odd we have that

$$\begin{aligned} \text{er}(n, k) &= \text{er}(n-2, k-2) + \text{er}(n-2, k-1) + \text{er}(n-2, k) \\ &\leq \text{frst}(n-2, k-2) + \text{frst}(n-2, k-1) + \text{frst}(n-2, k) \\ &= \text{frst}(n, k). \end{aligned}$$

Similarly, when  $k$  is even we have

$$\begin{aligned} \text{er}(n, k) &= \text{er}(n-2, k-2) + \text{er}(n-2, k-1) + \text{er}(n-2, k) \\ &\leq \text{frst}(n-2, k-2) + \text{frst}(n-2, k-1) + \text{frst}(n-2, k) \\ &\leq \text{frst}(n-1, k-1) + \text{frst}(n-2, k-1) + \text{frst}(n-2, k) \\ &= \text{frst}(n-1, k-1) + \text{frst}(n-1, k) \\ &= \text{frst}(n, k), \end{aligned}$$

where the second inequality follows from Lemma 3.5. These two cases complete the induction hypothesis.  $\square$

See Table 1 to compare the values of  $\text{frst}(n, k)$  and  $\text{er}(n, k)$  for  $n \leq 10$ .

## 4 Concluding remarks

Is it possible to find a  $q$ -(1+ $q$ )-analogue of the Gaussian coefficient which has the smallest possible index set? We believe that our analogue is the smallest, but cannot offer a proof of a minimality.



**Theorem 4.2.** *The distributive lattice  $\Omega_{n,k}$  has the decomposition*

$$\Omega_{n,k} = \bigcup_{v \in \Omega_{n,k}^r} \Omega_{r,1}^{b_1(v)} \times \Omega_{r,2}^{b_2(v)} \times \cdots \times \Omega_{r,\lfloor r/2 \rfloor}^{b_{\lfloor r/2 \rfloor}(v)}.$$

**Corollary 4.3.** *The  $q$ -binomial is given by*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{v \in \Omega_{n,k}^r} q^{\text{inv}(v)} \cdot \begin{bmatrix} r \\ 1 \end{bmatrix}_q^{b_1(v)} \cdot \begin{bmatrix} r \\ 2 \end{bmatrix}_q^{b_2(v)} \cdots \begin{bmatrix} r \\ \lfloor r/2 \rfloor \end{bmatrix}_q^{b_{\lfloor r/2 \rfloor}(v)}.$$

The least complicated case is when  $r = 3$ , where only one term appears in the above poset product. This term is  $\Omega_{3,1}$  which is the three element chain  $C_3$ . The associated Gaussian coefficient is  $1 + q + q^2$ . Thus Corollary 4.3 could be called a  $q$ -( $1 + q + q^2$ )-analogue in the case of  $r = 3$ . As an example, we have

$$\begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = 1 + q \cdot (1 + q + q^2)^2 + q^4 \cdot (1 + q + q^2)^2 + q^9.$$

On a poset level this is a decomposition of  $J(C_3 \times C_3)$  into two one-element posets of rank 0 and rank 9, and two copies of  $C_3 \times C_3$ , where one has its minimal element of rank 1 and the other of rank 4.

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